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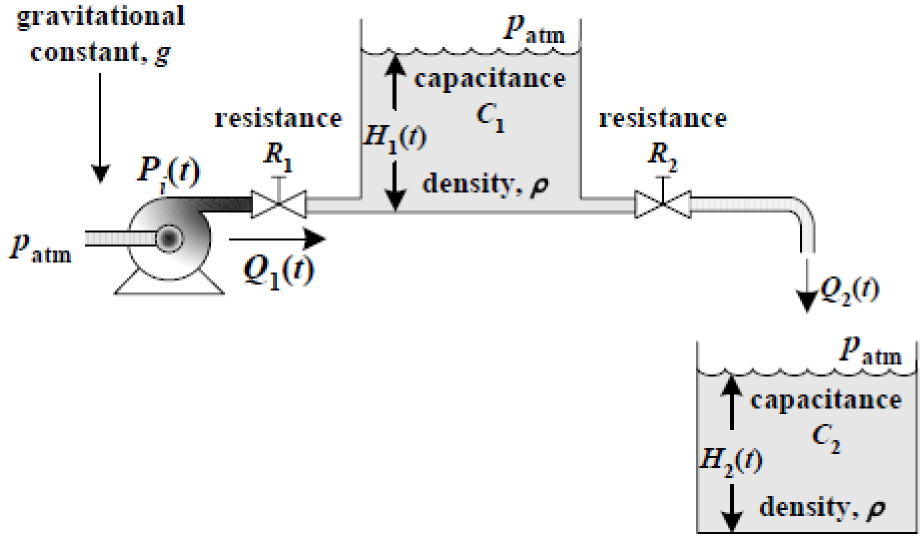

HOMEWORK #7

LIQUID-LEVEL SYSTEMS

TWO CASCADED LIQUID TANKS-PRESSURE SOURCE

ME 411: System Modeling Analysis and Control

Fall 2010

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November 18, 2010

1.0 <u>Initial steady state</u>:

At initial steady state (@ t =0), we have

$$\begin{cases} P_i(0) = \overline{P_i} \\ H_1(0) = \overline{H_1} \\ Q_1(0) = \overline{Q_1} \\ H_2(0) = \overline{H_2} \\ Q_2(0) = \overline{Q_2} \end{cases}$$

PUMP

$$\frac{\overline{P_i} + P_{atm}}{\rho g} - \frac{\rho g \overline{H_1} + P_{atm}}{\rho g} = R_1 \overline{Q_1}$$

$$\frac{\overline{P_i} + \cancel{P_{atm}} - \rho g \overline{H_1} - \cancel{P_{atm}}}{\rho g} = R_1 \overline{Q_1}$$

$$\frac{\overline{P_i}}{\rho g} - \overline{H_1} = R_1 \overline{Q_1} ------(1)$$

OULET

$$\frac{\rho g \overline{H_1} + P_{atm}}{\rho g} - \frac{P_{atm}}{\rho g} = R_2 \overline{Q_2}$$

$$\frac{\rho g \overline{H_1} + \cancel{P_{atm} - P_{atm}}}{\rho g} = R_2 \overline{Q_2}$$

$$\overline{H_1} = R_2 \overline{Q_2} ----- (2)$$

Tank2

$$\overline{H_2} = is\ independent\ of\ the\ fluid\ flow\ system\ \ as\ \ it\ is\ not\ link\ a\ resistance.$$

At any given instance the liquid height level will increase as the tank has only inlet no outlet and hence the value has no steady state or static equilibrium.

2.0 Transfer functions and governing equations:

The equation of the system as t > 0.

PUMP

$$\frac{\overline{P_i} + p_i(t) + P_{atm}}{\rho g} - \frac{\rho g\left(\overline{H_1} + h_1(t)\right) + P_{atm}}{\rho g} = R_1\left(\overline{Q_1} + q_1(t)\right)$$

$$\frac{\overline{P_i} + p_i(t) + \cancel{P_{atm}} - \rho g\left(\overline{H_1} + h_1(t)\right) - \cancel{P_{atm}}}{\rho g} = R_1\left(\overline{Q_1} + q_1(t)\right)$$

$$\frac{\overline{P_i} + p_i(t)}{\rho g} - \overline{H_1} - h_1(t) = R_1\overline{Q_1} + R_1 q_1(t)$$

$$\cancel{\left(\frac{\overline{P_i}}{\rho g} - \overline{H_1}\right)} + \left(\frac{p_i(t)}{\rho g} - h_1(t)\right) = \cancel{R_1\overline{Q_1}} + R_1 q_1(t) - - - from\ (eqn\ 1)$$

$$\left(\frac{p_i(t)}{\rho g} - h_1(t)\right) = R_1 q_1(t)$$

$$q_1(t) = \frac{1}{R_1}\left(\frac{p_i(t)}{\rho g} - h_1(t)\right) \rightarrow by\ Laplace\ transformation$$

$$Q_1(s) = \frac{1}{R_1}\left(\frac{P_i(s)}{\rho g} - H_1(s)\right) - - - - (3)$$

OULET

$$\frac{\rho g\left(\overline{H_1} + h_1(t)\right) + P_{atm}}{\rho g} - \frac{P_{atm}}{\rho g} = R_2\left(\overline{Q_2} + q_2(t)\right)$$

$$\frac{\rho g\left(\overline{H_1} + h_1(t)\right) + \cancel{P_{atm} - P_{atm}}}{\rho g} = R_2\left(\overline{Q_2} + q_2(t)\right)$$

$$\cancel{\overline{H_1}} + h_1(t) = \cancel{R_2\overline{Q_2}} + R_2 q_2(t) - - -\ from\ eqn\ 2$$

$$q_2(t) = \frac{h_1(t)}{R_2} \rightarrow by\ Laplace\ transformation$$

$$Q_2(s) = \frac{H_1(s)}{R_2} - - - - - - (4)$$

Tank 1

Compliance is given by

$$C_1 = \frac{dV}{dH_1} \rightarrow dV = C_1 dH_1$$

Conservation of mass,

$$\cancel{\rho} C_1 dH_1 = \cancel{\rho}[Q_1(t) - Q_2(t)]dt$$

$$C_1 d\left(\overline{H_1} + h_1(t)\right) = \left[\overline{Q_1(t)} + q_1(t) - \overline{Q_2(t)} - q_2(t)\right]dt$$

$$\cancel{C_1 d\overline{H_1}} + C_1 dh_1(t) = \left(\cancel{\overline{Q_1(t)}} + q_1(t) - \cancel{\overline{Q_2(t)}} - q_2(t)\right) dt$$

$$\frac{C_1 dh_1(t)}{dt} = \frac{1}{R_1}\left(\frac{p_i(t)}{\rho g} - h_1(t)\right) - \frac{h_1(t)}{R_2}$$

$$C_1 \frac{dh_1(t)}{dt} + h_1(t)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{R_1}\frac{p_i(t)}{\rho g}$$

$$\frac{R_1 R_2}{R_1 + R_2} C_1 \frac{dh_1(t)}{dt} + h_1(t) = \frac{R_1 R_2}{R_1 + R_2}\frac{1}{R_1}\frac{p_i(t)}{\rho g} \rightarrow by\ Laplace\ transformation$$

$$\frac{R_1 R_2}{R_1 + R_2} C_1 s H_1(s) + H_1(s) = \frac{R_2}{R_1 + R_2}\frac{P_i(s)}{\rho g}$$

$$i)\ \frac{H_1(s)}{P_i(s)} = \frac{\frac{R_2}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2}{R_1+R_2} C_1 s + 1\right)} - - - - - (5)$$

And

$$\frac{Q_1(s)}{P_i(s)} = \frac{1}{R_1}\left(\frac{1}{\rho g} - \frac{\frac{R_2}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2}{R_1+R_2} C_1 s + 1\right)}\right) - - - - from\ (3) and\ (5)$$

$$\frac{Q_1(s)}{P_i(s)} = \left(\frac{1}{\rho g R_1} - \frac{1}{R_1}\frac{\frac{R_2}{\rho g(R_1+R_2)}}{\frac{R_1 R_2 C_1 s + R_1 + R_2}{R_1+R_2}}\right)$$

$$\frac{Q_1(s)}{P_i(s)} = \left(\frac{1}{\rho g R_1} - \frac{1}{\rho g R_1}\frac{R_2}{(R_1 R_2 C_1 s + R_1 + R_2\ )}\right)$$

$$ii)\ \frac{Q_1(s)}{P_i(s)} = \left(\frac{\frac{R_2 C_1 s}{\rho g} + \frac{R_1}{\rho g}}{(R_1 R_2 C_1 s + R_1 + R_2\ )}\right) \rightarrow \left(\frac{\frac{R_2 C_1 s}{\rho g(R_1+R_2)} + \frac{1}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2 C_1 s}{R_1+R_2}\right) + 1\ }\right) - - - -(6)$$

Tank 2

Compliance is given by

$$C_2 = \frac{dV}{dH_2} \rightarrow dV = C_2 dH_2$$

Conservation of mass,

$$C_2 dh_2 = [q_2(t)]dt$$

$$C_2 dh_2 = \left[\frac{h_1(t)}{R_2}\right] dt$$

$$C_2 \frac{dh_2}{dt} = \frac{1}{R_2} h_1(t)$$

By Laplace transformation

$$C_2 s H_2(s) = \frac{1}{R_2} H_1(s) - - - - from\ (5)$$

$$iii)\ \frac{H_2(s)}{P_i(s)} = \frac{1}{R_2 C_2 s}\left(\frac{\frac{R_2}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2}{R_1+R_2} C_1 s + 1\right)}\right) \rightarrow \left(\frac{\frac{R_2}{\rho g(R_1+R_2)}}{\frac{R_1 R_2^2}{R_1+R_2} C_1 C_2 s^2 + R_2 C_2 s}\right) - - - - (7)$$

Again from equations (4) and (5)

$$iv)\ \frac{Q_2(s)}{P_i(s)} = \frac{1}{R_2} \frac{\frac{R_2}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2}{R_1+R_2} C_1 s + 1\right)} \rightarrow \frac{\frac{1}{\rho g(R_1+R_2)}}{\left(\frac{R_1 R_2}{R_1+R_2} C_1 s + 1\right)} - - - - - - (8)$$

Case	Differential Equations	Order
$\frac{H_1(s)}{P_i(s)}$	$\frac{R_1 R_2}{R_1 + R_2} C_1 \frac{d}{dt} h_1(t) + h_1(t) = \frac{R_2}{R_1 + R_2} \frac{1}{\rho g} p_i(t)$	First
$\frac{Q_1(s)}{P_i(s)}$	$\frac{R_1 R_2 C_1}{R_1 + R_2} \frac{d}{dt} q_1(t) + q_1(t) = \frac{R_2 C_1}{\rho g(R_1 + R_2)} \frac{d}{dt} p_i(t) + \frac{1}{\rho g(R_1 + R_2)} p_i(t)$	First
$\frac{H_2(s)}{P_i(s)}$	$\frac{R_1 R_2}{R_1 + R_2} C_1 C_2 \frac{d^2}{dt} h_2(t) + C_2 \frac{d}{dt} h_2(t) = \frac{1}{\rho g(R_1 + R_2)} p_i(t)$	Second
$\frac{Q_2(s)}{P_i(s)}$	$\frac{R_1 R_2}{R_1 + R_2} C_1 \frac{d}{dt} q_2(t) + q_2(t) = \frac{1}{\rho g(R_1 + R_2)} p_i(t)$	First

3.0 System parameters:

Case	Time constant (τ)	Static Sensitivity (K)	Damping constant (ζ)	Natural frequency (ω_n)	Damped frequency (ω_d)
$\frac{H_1(s)}{P_i(s)}$	$\frac{R_1 R_2 C_1}{R_1 + R_2}$	$K_0 = \frac{R_2}{R_1 + R_2} \frac{1}{\rho g}$	NA	NA	NA
$\frac{Q_1(s)}{P_i(s)}$	$\frac{R_1 R_2 C_1}{R_1 + R_2}$	$K_0 = \frac{1}{\rho g (R_1 + R_2)}$ $K_1 = \frac{R_2 C_1}{\rho g (R_1 + R_2)}$	NA	NA	NA
$\frac{H_2(s)}{P_i(s)}$	-	$K_0 = \infty$	0	0	0
$\frac{Q_2(s)}{P_i(s)}$	$\frac{R_1 R_2 C_1}{R_1 + R_2}$	$K_0 = \frac{1}{\rho g (R_1 + R_2)}$	NA	NA	NA

For second order system:

2^{nd}-order system dynamic equation: $$a_0 \frac{d^2 c(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_2 c(t) = b_0 \frac{dr(t)}{dt} + b_1 r(t)$$

or $$\frac{d^2 c(t)}{dt^2} + 2\zeta\omega_n \frac{dc(t)}{dt} + \omega_n^2 c(t) = \omega_n^2 K_1 \frac{dr(t)}{dt} + \omega_n^2 K r(t)$$

where $r(t)$ = reference input (*or* excitation *or* set point *or* desired value)
$c(t)$ = controlled output (*or* response)
a_0, a_1, a_2, b_0, b_1 = system parameters

$\omega_n = \sqrt{\frac{a_2}{a_0}}$ = (undamped) natural frequency

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$ = damping ratio

also $\omega_n = \omega_n \sqrt{1-\zeta^2}$ = damped natural frequency

$K = \frac{b_1}{a_2}$ = static sensitivity (*or* gain)

$K_1 = \frac{b_0}{a_2}$

However, in the physical world, coming up with the natural frequency equaling zero. And when this is the case, we know that T = 2*pi/omega, which says that T = infinity, so it never makes a complete period of getting back to where it started. So when we see a negative number for the natural frequency, assume that in the perfect world with no air drag and friction, the period is just infinity as when omega is zero and infinite gain.

Case	*Transfer Function*	*Zeros*	*Poles*
$\frac{H_1(s)}{P_i(s)}$	$\frac{\frac{R_2}{\rho g(R_1+R_2)}}{\frac{R_1 R_2 C_1 s}{R_1+R_2}+1}$		$-\frac{R_1+R_2}{C_1 R_1 R_2}$
$\frac{Q_1(s)}{P_i(s)}$	$\frac{\frac{R_2 C_1 s}{\rho g(R_1+R_2)}+\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1 R_2 C_1 s}{R_1+R_2}+1}$	$-\frac{1}{C_1 R_2}$	$-\frac{R_1+R_2}{C_1 R_1 R_2}$
$\frac{H_2(s)}{P_i(s)}$	$\frac{\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1 R_2}{R_1+R_2} C_1 C_2 s^2 + C_2 s}$		$-\frac{R_1+R_2}{C_1 R_1 R_2}$
$\frac{Q_2(s)}{P_i(s)}$	$\frac{\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1 R_2 C_1 s}{R_1+R_2}+1}$		$-\frac{R_1+R_2}{C_1 R_1 R_2}$

4.0 Steady state response:

Steady-state step responses. If the change in pressure source $p_i(t)$ is a step function:

$$p_i(t) = \bar{p}_i \cdot 1(t) = \begin{cases} \bar{p}_i & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

Using the final value theorem,

$$\boldsymbol{For}, f_{0ss} = \lim_{t \to 0} f_0(t) = \lim_{s \to 0} sF_0(s)$$

Case	*Transfer Function*	*Steady state*	*1% of steady state*
$\frac{H_1(s)}{P_i(s)}$	$\dfrac{\frac{R_2}{\rho g(R_1+R_2)}}{\frac{R_1R_2C_1s}{R_1+R_2}+1}$	$H_{1ss} = \frac{R_2}{\rho g(R_1+R_2)}\bar{p}_i$	$5\frac{R_1R_2C_1}{R_1+R_2}$
$\frac{Q_1(s)}{P_i(s)}$	$\dfrac{\frac{R_2C_1s}{\rho g(R_1+R_2)}+\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1R_2C_1s}{R_1+R_2}+1}$	$Q_{1ss} = \frac{1}{\rho g(R_1+R_2)}\bar{p}_i$	$5\frac{R_1R_2C_1}{R_1+R_2}$
$\frac{H_2(s)}{P_i(s)}$	$\dfrac{\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1R_2}{R_1+R_2}C_1C_2s^2+C_2s}$	$H_{2ss} = infinity$	Never reach a static equilibrium, no steady state always increase.
$\frac{Q_2(s)}{P_i(s)}$	$\dfrac{\frac{1}{\rho g(R_1+R_2)}}{\frac{R_1R_2C_1s}{R_1+R_2}+1}$	$Q_{2ss} = \frac{1}{\rho g(R_1+R_2)}\bar{p}_i$	$5\frac{R_1R_2C_1}{R_1+R_2}$